Due July 1, 11:59pm on Gradescope.

The following are warm-up exercises and are *not* to be turned in. You may treat these as extra practice problems.

2.3.20, 2.3.34, 2.3.35, 2.3.45, 2.5.10, 2.5.12, 2.5.22, 2.5.28, 2.5.34, 4.1.8, 4.1.41, 4.1.42, 4.1.47.

Turn in the following exercises. Remember to carefully justify every statement that you write, and to follow the style of proper mathematical writing. You may cite any result proved in the textbook or lecture, unless otherwise mentioned. Each problem is worth 10 points with parts weighted equally, unless otherwise mentioned.

1. **(5 points)** 2.3.46(b). You should also be aware of the result of 2.3.46(a), but it's not necessary to write it up.
2. Let $f : A \to B$ be a function between nonempty sets. Below, $f^{-1}(C)$ for a subset $C \subseteq B$ means the preimage of the subset C under f [see 2.3.46]. In particular, do not assume that f is bijective.
   (a) Prove that the following are equivalent: 1) f is injective. 2) f has a left inverse: there is a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$. 3) For *every* subset $A' \subseteq A$, $A' = f^{-1}(f(A'))$. [Hint: prove that statement 1 is equivalent to statement 2, then that statement 1 is equivalent to statement 3.]
   (b) Prove that the following are equivalent: 1) f is surjective. 2) f has a right inverse: there is a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. 3) For every subset $B' \subseteq B$, $B' = f(f^{-1}(B'))$.
3. For each part, construct (with proof) a function from $(0, 1)$ to $[0, 1)$ that is:
   (a) **(2 points)** neither injective nor surjective.
   (b) **(2 points)** injective but not surjective.
   (c) **(6 points)** bijective. [Remark: this shows that $(0, 1)$ and $[0, 1)$ have the same cardinality, even though the latter set has "one extra point," namely 0.]
4. (a) 2.5.40.
   (b) Using part (a) or otherwise, show that the set S of all functions from $\mathbf{N}$ to the 2-element set $\{0, 1\}$ is uncountable. [Hint: one method is to construct a bijection from S to a more familiar set.]

5. We define a complex number $z \in \mathbf{C}$ to be *algebraic* if it is the root of some monic polynomial equation
$$x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$$
of positive degree $n \geq 1$, where the a_i's are all rational numbers. If $z \in \mathbf{C}$ is not algebraic, then we call it *transcendental*. Prove that the set of all algebraic numbers is countable, and from this deduce that the set of transcendental numbers is uncountable. [Hint: you may assume without proof that any polynomial of degree $n \geq 1$ has at most n distinct complex roots.]

6. Compute the following. Any single-step explicit calculation involving a power greater than a 5th power should be justified (the idea is that you should do these without the help of a calculator, so you can only perform small calculations by hand, but you get a bit of leeway so as to not make things overly tedious).
   (a) Find the unique nonnegative integer a in $[0, 42]$ such that $6^{2^{100}+2} + 7^{2^{101}+1} \equiv a \bmod 43$.
   (b) Find the unique nonnegative integer b in $[0, 49]$ such that $11^{9^{7^{5^{3^{1}}}}} \equiv b \bmod 50$.

7. (a) Prove that if n is a perfect square, then n is congruent to 0 or 1 mod 4.
   (b) Prove that $3^{1000001}$ is not the sum of two perfect squares. [Hint: part (a).]

8. Show that there are no solutions in positive integers to the equation $a^2 + b^2 = 3c^2$. [Hint: assuming for contradiction that there is a positive integer solution (x, y, z), use Problem 7 and mimic the idea of the proof by contradiction (via "infinite descent") that $\sqrt{2}$ is irrational.]

9. **(Bonus problem, 10 points)** Find all solutions (x, y) in nonnegative integers to
$$2^x = 3^y + 7,$$
or prove that none exist. In particular, you should provide a list of solutions (possibly empty) to the given equation, and prove that there are no others.